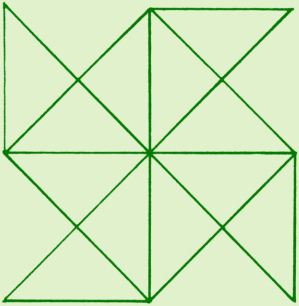
THINGS of science



MATHEMATICAL PAPER FOLDING

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MATHEMATICAL PAPER FOLDING

Geometric shapes, such as triangles, rectangles, polygons and circles, are familiar to all of you. Elementary concepts related to these shapes and forms have been known since the time of Euclid.

These concepts and spatial relationships can be more clearly understood and visualized in many cases by paper folding and paper models. With the materials in this unit you will be able to demonstrate many of the geometric principles and mathematical relationships.

First, identify the materials contained in this unit.

PROTRACTOR—Instrument to measure angles in degrees; yellow.

RULER—12-inch rule; green.

DIAGRAMS—Diagrams to serve as guides in paper folding.

THIN PAPER—Colored sheets for paper folding.

HEAVIER PAPER—Colored sheets for constructing models.

You will also need a pair of scissors and a pencil.

When performing the experiments in this unit, it is important that you be precise in your measurements and in fold-

ing to achieve accurate results.

For Experiments 1 through 26, use the lighter weight paper.

LINES

Experiment 1. Straight line. Cut one of your sheets of paper in half so that you have two sheets 8½ x 5½ inches in size.

A line has no width, only length. Two points always determine a straight line. On one of the half sheets of paper, make any two dots and label them A and B. Fold the sheet through these two points to form a straight line. The points and the straight line are elements of plane geometry, and the line you have creased through the two lines is the shortest distance between those two points.

Place a dot on each side of the crease, designating them C and D. If two straight lines intersect, they meet only at one point. Make a crease through C and D to demonstrate this.

Experiment 2. Locus of a point. Take the other half sheet and mark two points, P and Q, on this sheet. Fold the paper, placing point P exactly on top of point Q. Hold the two points together with one hand and crease the fold carefully with the other.

Does the fold form a straight line? Measure the distances from P and Q to various points of the fold line with your ruler. Are P and Q equally distant from all points on the fold?

The fold line is the locus (path) of all points equidistant from P and Q (Fig. 1).

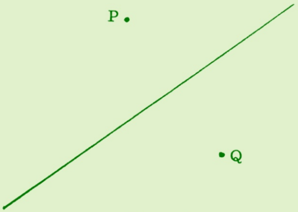

Fig. 1

Experiment 3. Parallel lines. Take another half sheet and crease it exactly in half. Make another crease exactly parallel to this. Two parallel lines are equally distant from each other and will not meet.

Mark the parallel lines AB and CD. Make a diagonal crease across the two parallel lines designating the points of intersection P on line AB and Q on line CD.

Take your protractor and measure the angle QPB. Is it equal to angle PQD? Measure CQP. Is this angle equal to

any of the other angles? Is angle APQ equal to any of the other angles?

A transversal intersecting two parallel lines forms equal angles with them.

Make another fold across the two parallel lines at a slightly different angle and measure the angles formed at the intersections. Do you find the same relationships as before?

Two line segments that meet in a point form an angle. Angles are classified according to their size. An angle of 90 degrees is a right angle. One of 180 degrees is a straight angle. An angle is acute when it measures less than 90 degrees and obtuse when it measures more than 90 degrees but less than 180 degrees (Fig. 2).

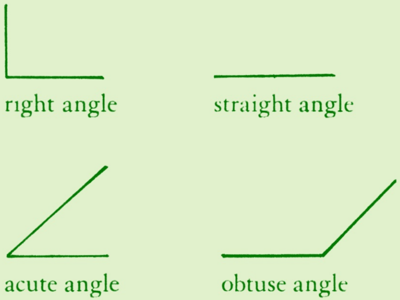

Fig. 2

TRIANGLES

Experiment 4. Take a sheet of paper and mark it with three widely spaced points, indicating them as A, B and C. (See large sheet of diagrams: Diagram 1a.)

Make a crease from A to B, B to C and C to A. Cut along the creases to form a triangle.

The simplest figure having straight sides only that can completely enclose a plane is a triangle. Every triangle has three straight sides, three angles and three vertices or points where the sides meet.

When described in terms of their angles, there are three types of triangles, acute, obtuse and right. In an acute triangle, all angles are less than 90 degrees. An obtuse triangle has two acute angles and one obtuse angle. A right triangle has one angle that is 90 degrees.

Experiment 5. What is the sum of the angles of a triangle? This can be easily found by paper folding.

Take the triangle ABC you have just cut out and make a fold through the vertex C perpendicular to the base, AB. Mark the point at which the perpendicular intersects the base D. The fold makes two 90-degree angles with the base. The line you have made from the base to the

vertex is the altitude of the triangle (Diagram 1a).

Bring vertex C down to meet point D (Diagram 1b). Bring vertices A and B to point D.

Do the three angles meet to form a straight line? Is the sum of the angles 180 degrees?

Experiment 6. What is the area of a triangle?

By folding the vertices of the triangle to meet at point D, you have formed two rectangles of equal size, one superimposed upon the other. You can observe this by lifting up the three vertices.

The area of a rectangle is the product of the length times the width. What is the area of your triangle?

Experiment 7. Cut out an obtuse triangle. Fold the vertices of the triangle down to the base in the same way as before. Is the sum of the angles in this triangle 180 degrees also?

You will find that the sum of the angles of a triangle is 180 degrees no matter what the shape of the triangle.

Triangles may be classified according to the relative lengths of their sides. If two sides are equal, the triangle is an isosceles triangle. If all three sides are equal, it is an equilateral triangle. If

each of the three sides is of a different length, it is a scalene triangle.

Experiment 8. Cut a square 5 by 5 inches in size from a half sheet of your paper. Check the angles with your protractor to be sure they are all exactly 90 degrees. Fold the square diagonally and you have two isosceles right triangles, the sides of the square forming the two equal sides. In an isosceles triangle, the angles opposite the equal sides are equal. Measure the angles with your protractor. Since the right angle is 90 degrees, each of the other two angles must measure 45 degrees.

Experiment 9. Construct an equilateral triangle.

Make a fold and measure off 5 inches for one side, designating the line, AB. Find the midpoint of AB (the point equidistant between A and B). Mark this point D. Now fold the sheet perpendicular to side AB at the midpoint, D, and crease the fold. Open up the paper again. With your ruler measure 5 inches from point A to the perpendicular fold. Mark this point C. Make a fold from point A to C and then make another fold from B to C. AC should be equal to BC.

Cut along the outside folds for your equilateral triangle ABC. All the angles

of an equilateral triangle are equal. Since the sum of the angles of a triangle is 180 degrees, how many degrees are there in each of the three angles? Check your answer with your protractor to see how accurate you have been in constructing your triangle.

Fold the equilateral triangle in half and you will have two right triangles with angles 90, 60 and 30 degrees (Diagram 2).

Cut the triangle along the perpendicular CD forming two right triangles. Bring the base AD of the triangle ADC in line with side AC, making a fold through angle CAD. Fold again, bringing point A to point C. Is AD one-half of AC?

In a triangle with angles measuring 90, 60 and 30 degrees, the longest side is always twice the length of the shortest side, or AC = 2 AD.

Experiment 10. Intersection of the altitudes of a triangle. Cut out a triangle of any shape from a half sheet of your paper and fold the altitudes from each of the three vertices to the opposite sides. Do the lines intersect at a common point (Diagram 3a)? Trace these lines with blue ink or pencil.

Experiment 11. Fold each of the angles of the triangle exactly in half and

crease the fold to the opposite base. The line of the fold indicates the bisector of the angle. Do these lines meet at a common point (Diagram 3a)? Trace these lines in red.

Experiment 12. Next fold the perpendiculars from the point of intersection of the bisectors to each of the three sides or bases. Are they all equal in length? Draw these lines in a third color (Diagram 3a).

Experiment 13. Repeat the three preceding experiments with an equilateral triangle cut from another half sheet of paper (Diagram 3b). Do all the folds meet at a common point? Measure the perpendiculars from the point of intersection to each side. Are they all of equal length? Do the folds bisect the sides and the angles?

The point at which these lines intersect is known as the center of symmetry.

Experiment 14. Two triangles having equal bases and equal altitudes have equal areas. Draw two triangles of any shape having equal bases and equal altitudes.

Fold them as you did the triangles in Experiment 6 and find the area of the triangles. Compare the rectangles formed. Are they equal in area?

Experiment 15. If two sides of one

triangle are parallel to the corresponding two sides of another triangle and straight lines can be drawn from an external point through their vertices, then the third sides of the triangles are also parallel to each other.

Take a sheet of your paper and fold it diagonally from corner to corner. Mark this line AB (Diagram 4). Fold the paper from point A to any point C above AB and from point A to any point D below AB. From any point E on AB, make a fold to any point F on line AC and G on line AD. These folds form the two sides of a triangle.

Fold lines parallel to EF and EG from any point H on line AB to lines AC and AD respectively. Designate these lines HI and HJ. Make a fold from F to G and from I to J. Are these two lines parallel (Diagram 4)?

POLYGONS

Experiment 16. Polygons are closed plane figures bounded by straight lines. A four-sided polygon can be a square, rectangle, parallelogram, or a trapezoid.

A parallelogram is a quadrilateral whose opposite sides are parallel. In a parallelogram, opposite angles are equal (Diagram 5a).

Area of a parallelogram. Cut out a parallelogram like the figure shown in Diagram 5a. Fold a right triangle on one end of the parallelogram. Cut off the triangle along the altitude formed and fit it against the other end of the parallelogram to form a rectangle. The area of the parallelogram is thus found to be equal to the product of its base and altitude (Diagram 5b).

Experiment 17. Any triangle is always of equal area to a parallelogram having a base equal to that of the triangle and an altitude one-half as great as that of the triangle.

Take a half sheet of paper and cut out a triangle with a base 4 inches and altitude 8 inches.

Cut out a parallelogram with a base equal to that of the triangle (4 inches) and an altitude of 4 inches (half the height of the triangle).

Fold and cut the parallelogram to find its area as you did in Experiment 16. Find the area of the triangle as you did in Experiment 6. Do the areas seem to be about the same?

Experiment 18. Two parallelograms having equal bases and equal altitudes are of equal area. Cut out two different parallelograms having equal bases and

equal altitudes from a half sheet of your paper. Fold and cut them to find the areas of each (Diagrams 6a and 6b).

Place one rectangle over the other. Are they equal in dimensions?

Experiment 19. Diagonals of a parallelogram bisect each other. Cut out another parallelogram and fold the diagonals. Measure the lengths of the intersected segments with your ruler. Are they equal?

Experiment 20. The diagonals of a rhombus intersect at right angles.

A rhombus is an equilateral parallelogram. Cut out a parallelogram with equal sides and fold it along the diagonals (Diagram 7).

What kind of angles are formed at the intersection of the diagonals? Measure them with your protractor.

Experiment 21. Area of a rhombus. Mark the diagonals in your rhombus AB and CD and the point of intersection O. Cut along the diagonal CD and the segment AO.

Place side AD of the triangle AOD thus formed along side CB with A at C and AC of triangle AOC along side DB with A at D. What geometric figure does this produce? From this you can see that the area of the rhombus is the product of the length of the segment AO and the

diagonal CD, or ½AB x CD.

Experiment 22. Area of a square within a square. Cut out a square from a half sheet of paper and fold it in half both ways. Open up the paper and you will see four small squares. Designate the midpoint of the sides of the original square, A, B, C and D. Make folds from midpoint to midpoint—A to B, B to C, C to D and D to A. Note that these folds are along the diagonals of the small squares and that the vertices all meet at the center of the original square (Diagram 8). Open up the folds and you will see a square within the original square. What is the area of this square?

CIRCLE

Experiment 23. Note the parts of the circle shown in Diagram 9. Draw a circle on a half sheet with a compass, or use a round object such as a dish or glass as a guide. Cut out the circle and find its center by folding the circle in half on two different diameters. The point of intersection of the folds is the center of the circle. Mark this point O.

Fold the circle in half again and then fold a portion of the circle through the end point of the diameter. Label the end point A. Unfold the circle. You have

formed two equal chords. Designate the chords AB and AC (Diagram 10). Do equal chords have equal arcs?

Fold the radii BO and CO. Are the angles formed by AOB and AOC equal? Why are triangles formed by radii and a chord always isosceles? What other observations can you make regarding equal chords in a circle?

Experiment 24. The perpendicular bisector of a chord. Fold any chord AB. Fold the radii AO and BO (Diagram 11). Bisect the angle AOB by bringing A to B and creasing the paper along the radius CO. Is the bisector of the angle perpendicular to AB? Does the perpendicular bisector also bisect the chord AB? What other relationships do you find?

Experiment 25. Angles inscribed in a semicircle. Cut out another circle and fold it along any diameter, marking it AB. Fold a chord from endpoint A to any point C. Now make a fold from C to B (Diagram 12). What type of triangle is formed?

Repeat using another chord AD. Is this triangle a right triangle also?

ALGEBRAIC CONSTRUCTION

Experiment 26. Some algebraic equations can also be demonstrated by paper

folding. An example is the equation,

$$(a+b)^2 = a^2 + 2ab + b^2$$

Take a square piece of paper. Fold it down one side parallel to the edge. Fold it down an equal distance along an adjacent side (Diagram 13).

By cutting each section out and arranging the pieces according to the equation, you can prove the truth of the above expression.

GEOMETRIC MODELS

Experiment 27. Use the heavier paper for constructing your geometric models.

Construct an equilateral hexagon, or six-sided figure (Diagram 14).

Draw and cut out an equilateral triangle with vertices A, B and C from a half sheet of paper. Fold each of these to the center of the triangle and a hexagon is formed.

Experiment 28. Form an octagon, an eight-sided figure (Diagram 15). Take a square piece of paper and fold each corner to the center and crease. Unfold the paper and then fold it again, this time bisecting the eight angles formed by the edges of the original square and the inscribed square. Find the octagon.

Experiment 29. Make a six-pointed

star. Cut out a fairly large circle and locate its center. Fold a chord AB so that its arc touches the center of the circle. Crease the fold. Fold another chord BC to the center as before, allowing the arc of this chord to begin at B, and then fold a third chord AC. You will have three equal chords, forming a triangle. Fold each of the vertices of the triangle to the exact center of the circle and crease the fold. Unfold the paper and turn the circle over. You will see the outlines of a six-pointed star (Diagram 16).

Experiment 30. Polygon construction by tying knots.

Cut a half-inch wide strip of paper 11½ inches long and tie a simple knot carefully (Diagram 17). Crease it flat and cut off the excess ends. Can you see the pentagon formed?

Note that the knot also forms trapezoids, four-sided figures with only two sides parallel. How many trapezoids do you see? How do they compare?

Experiment 31. To construct a hexagon, cut a strip of paper ½ inch wide by 11½ inches long of one color and another strip of the same length and width from a different color. Loop the strips and draw in the knot as shown in Diagrams 18a and 18b. Crease the paper flat.

Experiment 32. Five-pointed star. Cut out a square and fold it along one diagonal to form two triangles with a common base, AC, and vertex B. Locate the midpoint D of the base. Divide side AB into thirds, marking the division points E and F (Diagram 19a).

Fold CD over AB so that side CD intersects AB at point E and side BC intersects AB at F (Diagram 19b). Designate the point of the fold on side BC as G. Now bring side GD to meet CD and form the crease FD (Diagram 19c).

Fold side AD under to meet side FD (19d) and crease along ED.

About ⅓ the distance from D to G (Diagram 19d), mark a point H. Cut along a straight line from H to F. Open up the folds of HFD and you have your five-pointed star.

THREE-DIMENSIONAL MODELS

Experiment 33. To construct a tetrahedron, or a figure consisting of four triangular faces, draw an equilateral triangle. Fold each of the vertices to the midpoint of the base opposite (Diagram 20). Bring the points of the three vertices of the triangles together and you have a model whose faces are four equilateral triangles.

Experiment 34. Follow Diagram 21

to construct an octahedron. Cut out the model and fold the equilateral triangles along the solid lines. Tape the free edges together where they meet. Count the faces.

Experiment 35. Construct a cube. Cut a square about 8½ x 8½ inches in size from a sheet of paper. Fold the paper along the two diagonals and crease. Open the paper up and turn it over. Now fold the square in half crosswise along one length creasing the fold. Open the paper up and then fold it into the shape shown in Diagram 22a, marking points A, B, C, D and E on the paper.

Fold points A and D down to point E and mark points F and G (Diagram 22b).

Turn the paper over and bring B and C down to E and mark points H and I (Diagram 22c) in the same way. Mark the point at the top O.

Fold the corners F and G so that they meet at the center (Diagram 22d). Turn the paper over and do the same with the corresponding corners H and I.

Fold the free corners A and D outward on the front (Diagram 22e). Do the same with the corresponding corners C and B on the reverse side. Fold points A and

D so that they meet at the center (Diagram 22f). Crease the folds. Do the same with points B and C on the other side. Note the triangles formed by these folds.

Tuck the triangles inside the convenient pockets just beneath them formed by F, G, H and I (Diagram 22g).

Now blow forcefully into the small hole found at O and inflate the cube (Diagram 22h).

Many other mathematical models may be constructed by paperfolding. If you wish to pursue the subject further, the references given below will be helpful.

Mathematical Models, H. Martyn Cundy and A. P. Rollett, Oxford University Press.

Paper Folding for the Mathematics Class, Donovan A. Johnson, National Council of Teachers of Mathematics, Washington, D. C.

Appreciation is expressed to the National Council of Teachers of Mathematics for their helpful suggestions in the preparation of this unit.

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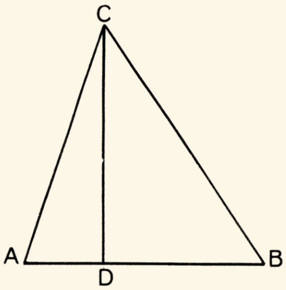
MATHEMATICAL PAPER FOLDING DIAGRAMS



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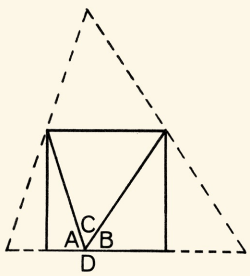

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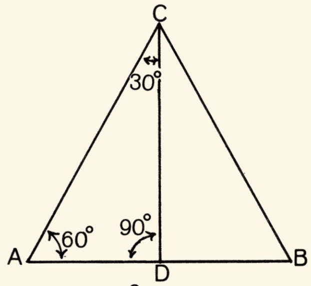

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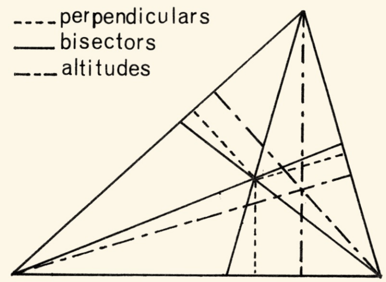

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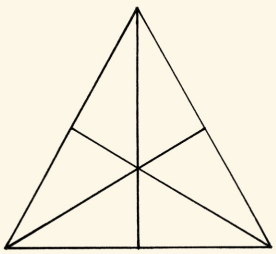
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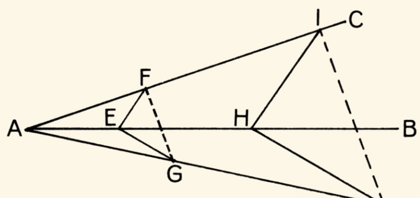

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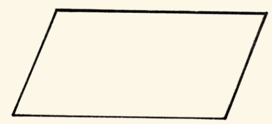
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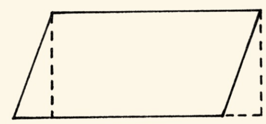
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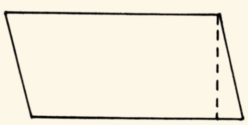
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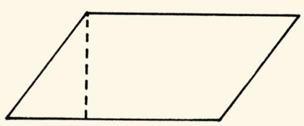
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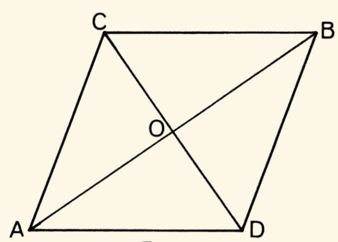

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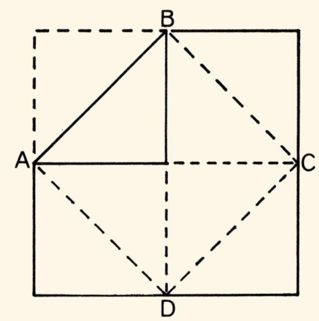

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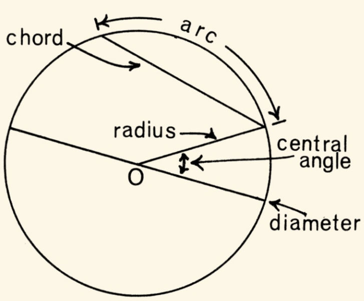

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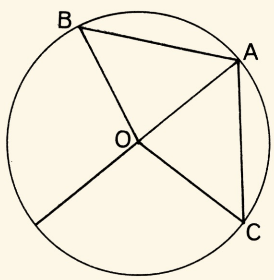

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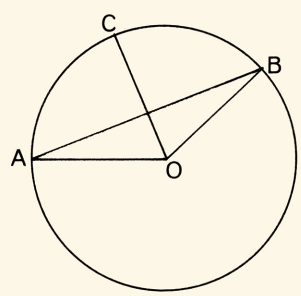

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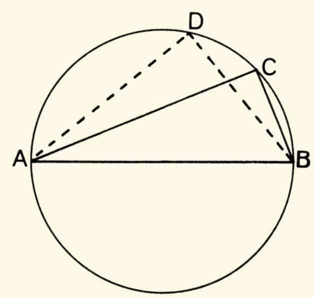

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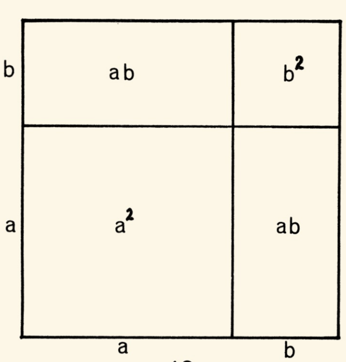

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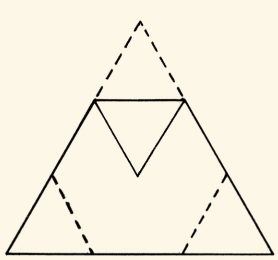
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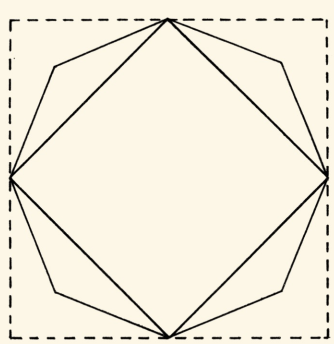
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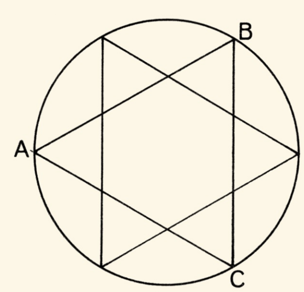

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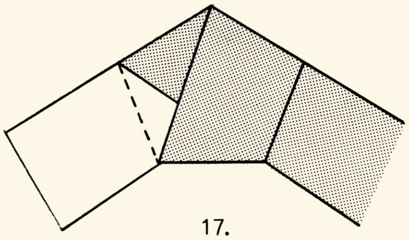

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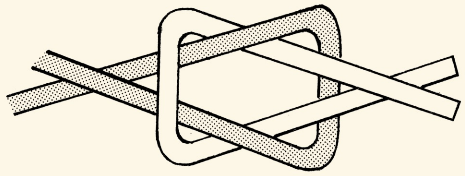

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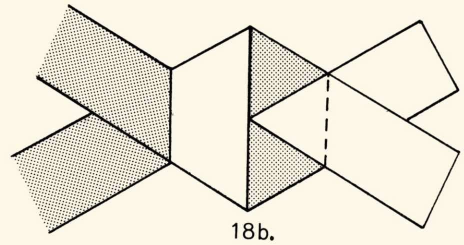

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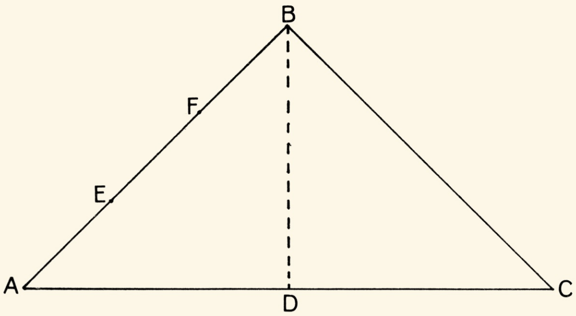

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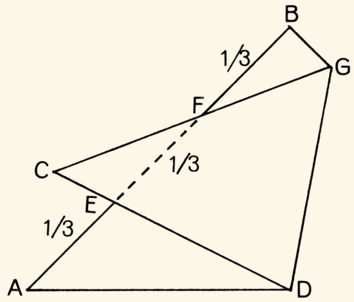

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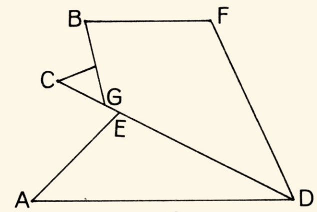

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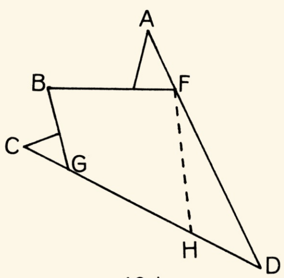
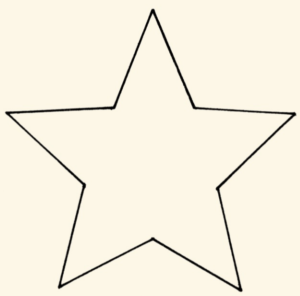


19d.



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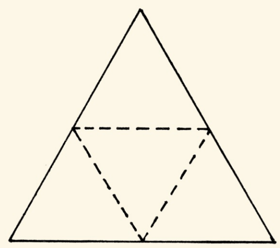

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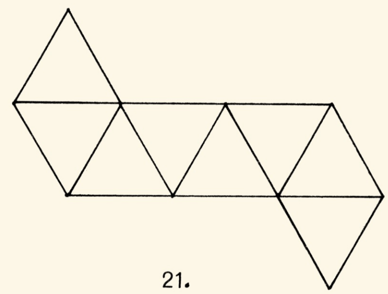

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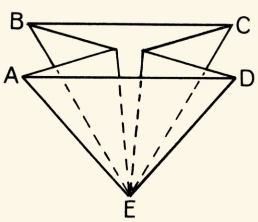

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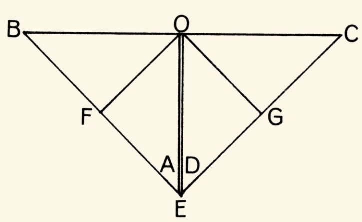

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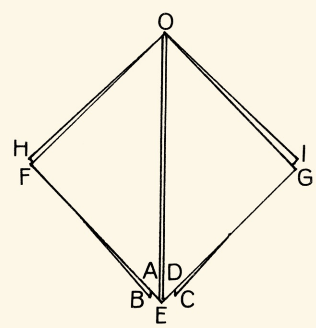

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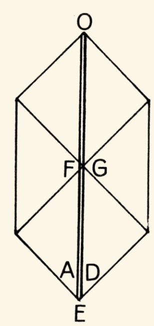

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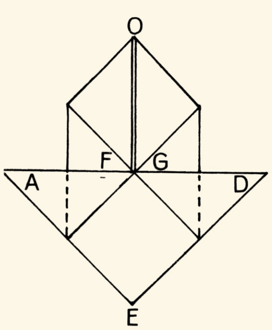

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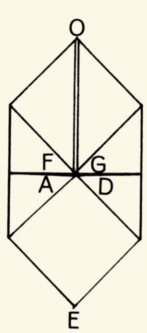

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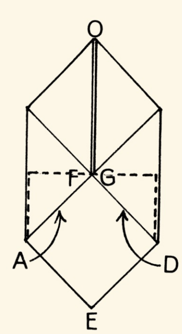

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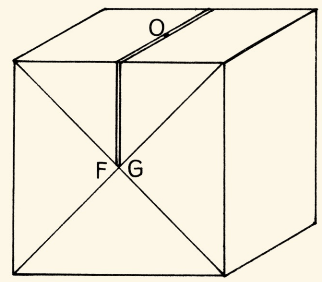

22h.